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Principal scheme of a deuteron EDM ring with a long spin coherence time. (Cancellation of the second-order perturbations in $\Delta\omega_a$)

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1. The basic idea.

The goal of this EDM Note is to show how we can cancel the second-order effects of betatron and synchrotron oscillations violating the main condition of our deuteron EDM experiment ,

$$\omega_a = \frac{e}{m}\left\{|a|B_V - \left[|a| + \left(\frac{m}{p}\right)^2\right]\beta E_R\right\} = 0, \quad \text{ideal case.} \tag{1.1}$$

The basic idea is the following. Consider, for example, the violation of (1.1) by horizontal betatron oscillations, $x(s)$, s is the longitudinal coordinate,

$$x_\beta(s) = A_x\sqrt{\beta_x(s)}\cos(\psi_x(s)+\delta_x). \tag{1.2}$$

(The subscript β distinguishes free betatron oscillations from closed orbits depending on $\Delta p/p$. We will omit this subscript when it will not lead to the reader's confusion.) In (1.2), I use the Handbook [1] expression for $x(s)$, p.49, section 2.1, formula (2), in which the dimension of A_x is $m^{1/2}$, not m. Any terms linear in x in (1.1) are averaged to zero over time, but the quadratic are not. As a result, we have there the horizontal pitch effects proportional to $\langle x^2\rangle$ and $\langle \vartheta_x^2\rangle \equiv \langle (dx/ds)^2\rangle$, summarized as

$$(\Delta\omega_a)_{\substack{betatron\\ horizont}} \equiv \Delta\omega_{a1} = a_1 A_x^2. \tag{1.3}$$

The factor a_1 depends on the lattice structure. (1.3) is an incoherent, individual perturbation of ω_a, different for particles having different amplitudes A_x. Therefore, it leads to a steady loss of the beam polarization P in time. Our goal is, of course, to prolong the coherence time as much as possible. So if we want to keep our beam polarized up to, say, 1 minute, while permitting the

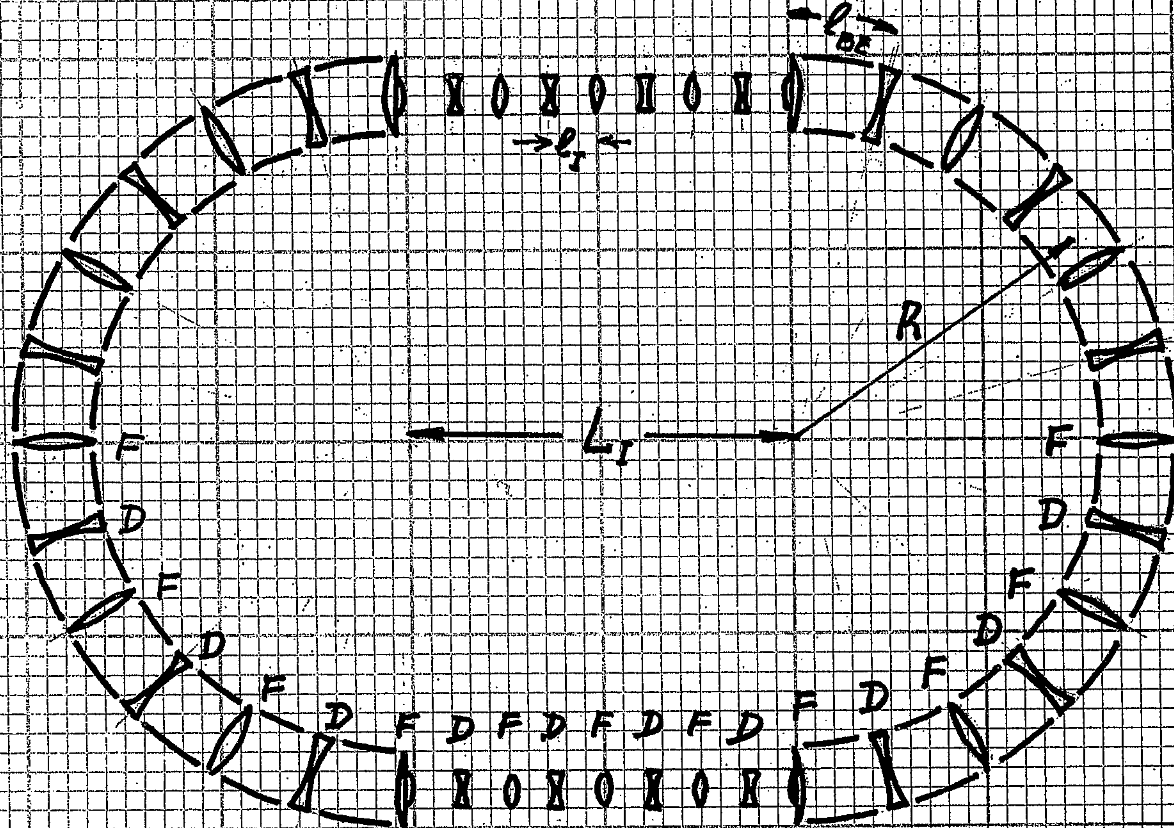


Fig. 1 The ring.

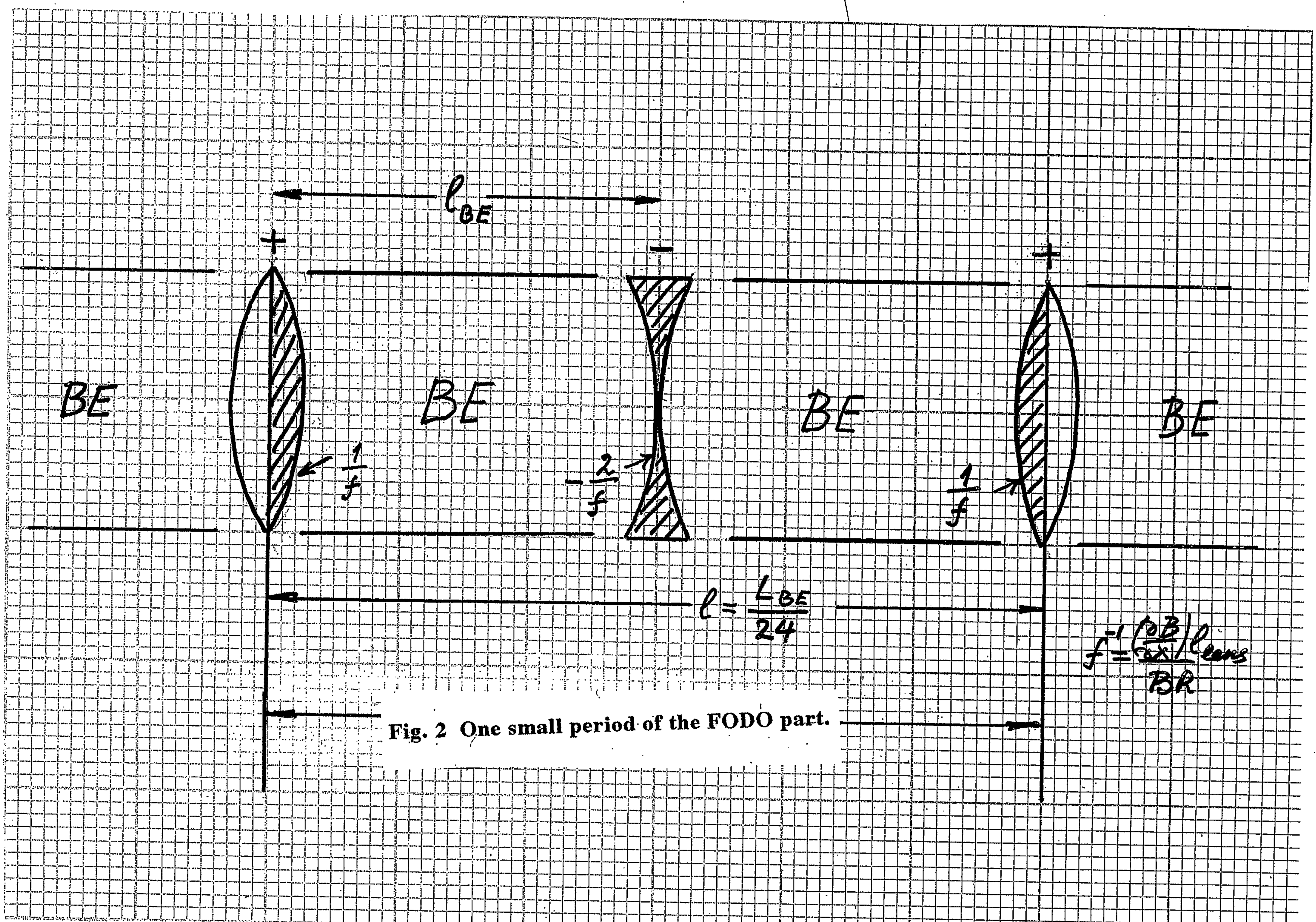


Fig. 2 One small period of the FODO part.

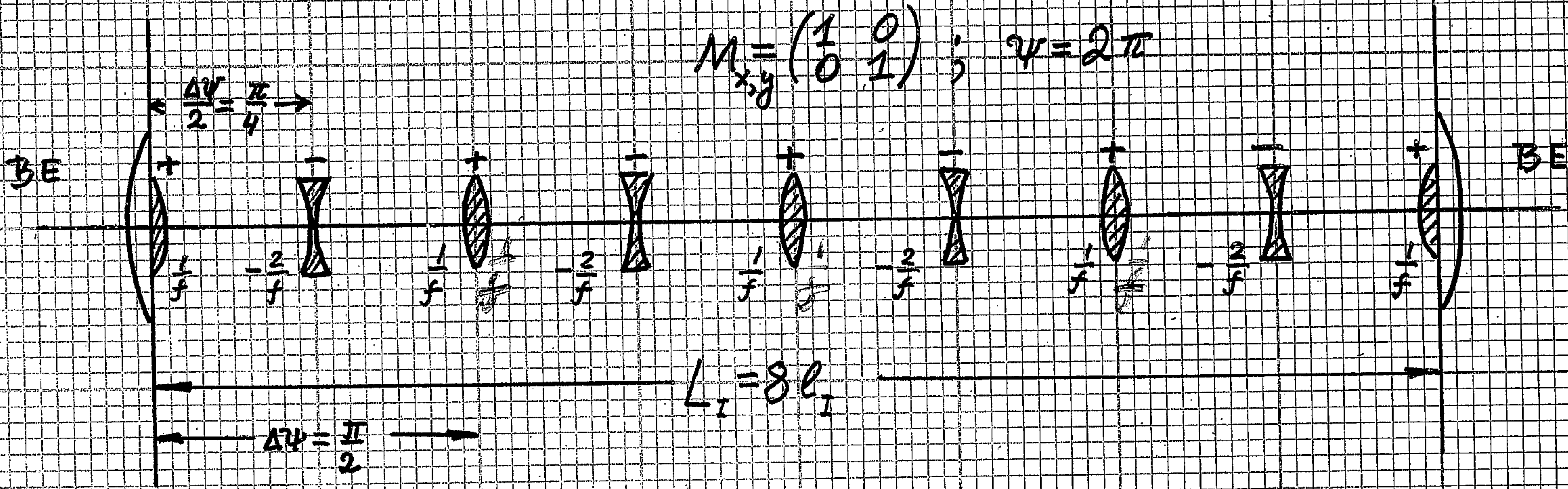


Fig. 3 Straight section.

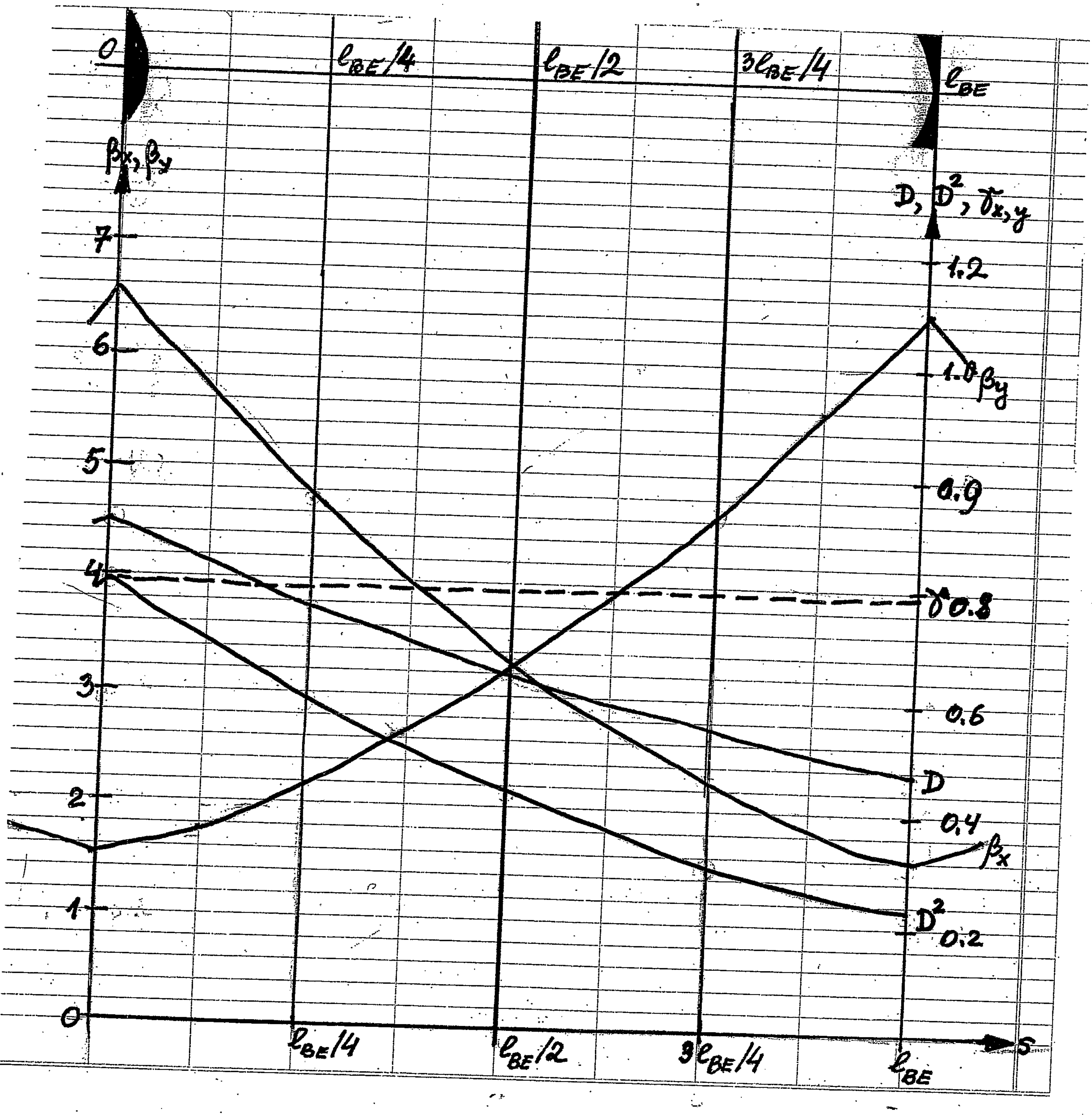
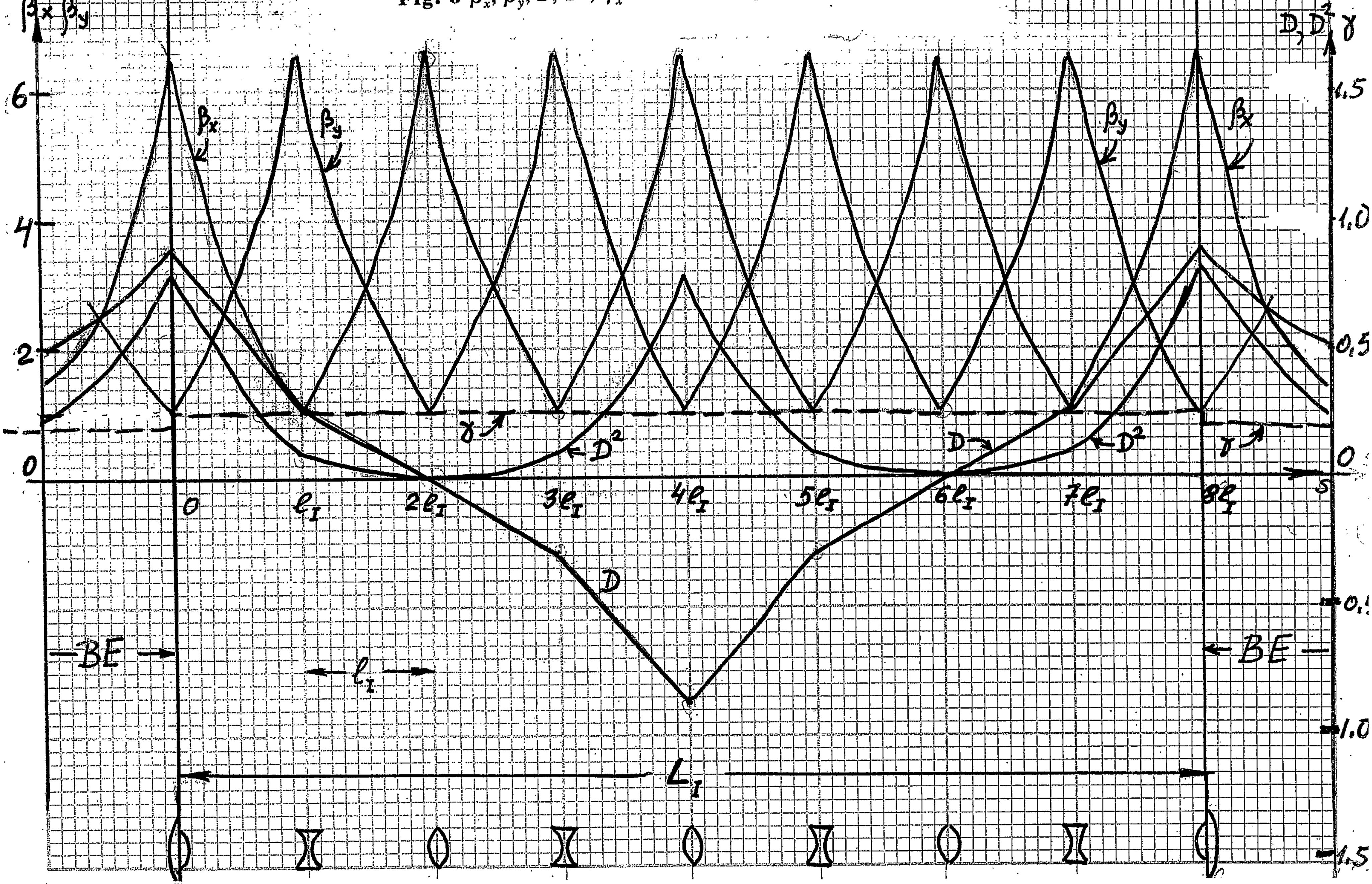


Fig. 5 β_x, β_y, D, D^2, γ_x **in the semicircle FODO part of the ring.**

Fig. 6 β_x, β_y, D, D^2, γ_x in the straight section.



2. Main elements of the lattice of a deuteron EDM ring.

The size of such a ring, Fig. 1, is more or less defined by an assumed magnitude of the radial electric field, E_R, and the desirable momentum of deuterons, p. Certainly, we need p>0.6 GeV/c. I have assumed the electric field at the equilibrium orbit, E=4 MV/m=1.3333×10^{-2} T, and the equilibrium deuteron momentum, p=0.788 GeV/c. This gives me the equilibrium orbit radius inside the BE sections, R=15 m, and the magnetic field there, B=0.2095 T. "BE" means the combination of the vertical magnetic field, B_V, and the radial electric field, E_R. At the ideal equilibrium orbit, condition (1.1) holds, where a=-0.143, m=1.8756 GeV. The related formulas I have used are

$$p = 0.3(B - E/\beta)R, \tag{2.1}$$

$$B = \frac{1}{|a|}\left[|a| + \left(\frac{m}{p}\right)^2\right]\beta E, \tag{2.2}$$

$$E = \frac{1}{0.3R}\times\frac{|a|p\beta}{\beta^2\left[|a| + \left(\frac{m}{p}\right)^2\right] - |a|}, \tag{2.3}$$

$$p = 0.3ER\left[\frac{|a| + \left(\frac{m}{p}\right)^2}{|a|}\beta - \frac{1}{\beta}\right], \tag{2.4}$$

$\beta = v/c = 1/\sqrt{1+(m/p)^2}$, p in GeV/c, B and E in T , R in meters.

Thus,

$\times 10^{-2}$

p=0.788, E=1.3333, R=15, B=0.2095, $\beta = 0.3873$, $\beta^2 = 0.15$, $(m/p)^2 = 5.6654$,

$$\gamma^2 = 1 + (p/m)^2 = 1.1765,\ \gamma = 1.0847,\ f_c = 0.9306\ \text{MHz},\ \frac{1}{2\pi}\frac{\text{e}}{\text{m}}|a|\text{B}_v = 144.34\ \text{kHz} \tag{2.5}$$

Any design of a deuteron EDM ring must obey the following physical conditions:

I. The ideal ring must be symmetric with respect to the clockwise (CW) and counterclockwise (CCW) movements of the deuterons. During the CCW runs, the sign of the magnetic field must be changed not only in the magnetic dipoles, but in all magnetic elements of the lattice shown in Figs. 2, 3. (In particular, a ring version designed for muons in [3], where the magnetic lens currents must be not changed, is not acceptable in the much more precise deuteron approximation, it is canceled on the average (meaning average in time) by introducing synchrotron oscillations of the particle momenta with the help of the corresponding RF cavities, $\Delta p/p = (\Delta p/p)_0 \cos(\omega_s t + \phi)$. However, synchrotron oscillations are not exactly linear. There are various quadratic terms in the synchrotron equations, different for different particles, and these terms shift the equilibrium momenta of these particles. As a result, the main off-momentum effect violating (1.1) is not zero, and is not linear:

$$\Delta p/p = \langle \Delta p/p \rangle + (\Delta p/p)_0 \cos(\omega_s t + \phi), \tag{1.7}$$

where $\langle \Delta p/p \rangle$ is shifted from zero value by second-order effects proportional to $\langle A_x^2 \rangle$, $\langle A_y^2 \rangle$, and $\langle (\Delta p/p)^2 \rangle$. Such a shift influences all three factors a_1, a_2, and a_3, so we can use it in our design to control (1.1) by manipulating synchrotron equilibrium, i.e., $\langle \Delta p/p \rangle$. The principal possibility of controlling all three factors by sextupoles arises from the fact that the full horizontal oscillations $x(s)$ in (1.4) contain both betatron and synchrotron oscillations,

$$x(s) = x_\beta(s) + D(p,s)\Delta p/p, \tag{1.8}$$

$x_\beta(s)$ from (1.2). (About function $D(p,s)$, see [1], p. 50.) Therefore, on the average,

$$\langle x^2 \rangle = \langle x_\beta^2 \rangle + D^2 \langle (\Delta p/p)^2 \rangle. \tag{1.9}$$

(The uncounted linear term, $2Dx_\beta(p/p)$ plays an important role in the problem of the free betatron oscillations' chromaticity. But it is not our main concern here.) Thus, on the average, field (1.4) of sextupole #k is

$$\langle B_k \rangle = B_k'' \left[\langle x_\beta^2 \rangle_k - \langle y_\beta^2 \rangle_k + D_k^2 \langle (\Delta p/p)^2 \rangle \right]. \tag{1.10}$$

In this Note I show that, with the help of the sextupoles, the proposed accuracy of $\omega_a = 0$ is possible in principle. But the calculations for the final design need much more work. For example, I have used here a thin lens approximation, which is not exactly realistic. I have done so because such an approximation is very transparent, almost all effects can be represented analytically, and many preparatory formulas can be verified by [1]. A number of higher-order effects which are needed in order to know a_1, a_2, and a_3 more precisely—with the accuracy 10^{-3}—are not taken into account here. Also, in order not to complicate the main subject of this Note, I have not included the acceleration of particles by the radial electric field into the calculations.

We now need to take into account the sextupole fields, see eqs. (1.4), (1.10) above. The shortest way to calculate α_2 is to use formula (2) in [1], p. 263: If a particle passing a very short area $\Delta s = l_{si}$ of the magnetic perturbation, ΔB_i, gets the same kick (angle deflection) θ_i during every revolution, then, on the average, its closed orbit length is changed as

$$\Delta L = \theta_i D(s_i). \tag{4.13}$$

(This formula is consistent also with the Hamiltonian (44) represented in [1], p. 70.) The kick produced by a perturbation ΔB equals $\theta = -\Delta B l_s / BR$; in the case of a sextupole, $\Delta B = (B'' D^2 l_s / 2BR)(\Delta p / p)^2$; therefore,

$$\alpha_2 = -\sum \frac{B_i'' D^3(s_i) l_{si}}{2BRL}, \qquad B'' \equiv \partial^2 B / \partial x^2. \tag{4.14}$$

3. The relevant lattice formulas and parameters.

First of all, we need to specify the B and E fields in the BE section. If the electric plates are infinite straight vertical plates, as we assume here, then the ideal electric field

$$E = E_R(x) = \frac{E_0 R_0}{R} = \frac{E_0}{(1+x/R_0)}. \tag{3.1}$$

It is not unreasonable to design the ideal magnetic field of the BE sections so that it has similar behavior in the horizontal plane:

$$B(y=0) = B_V(x) = \frac{B_0 R_0}{R} = \frac{B_0}{(1+x/R_0)}. \tag{3.2}$$

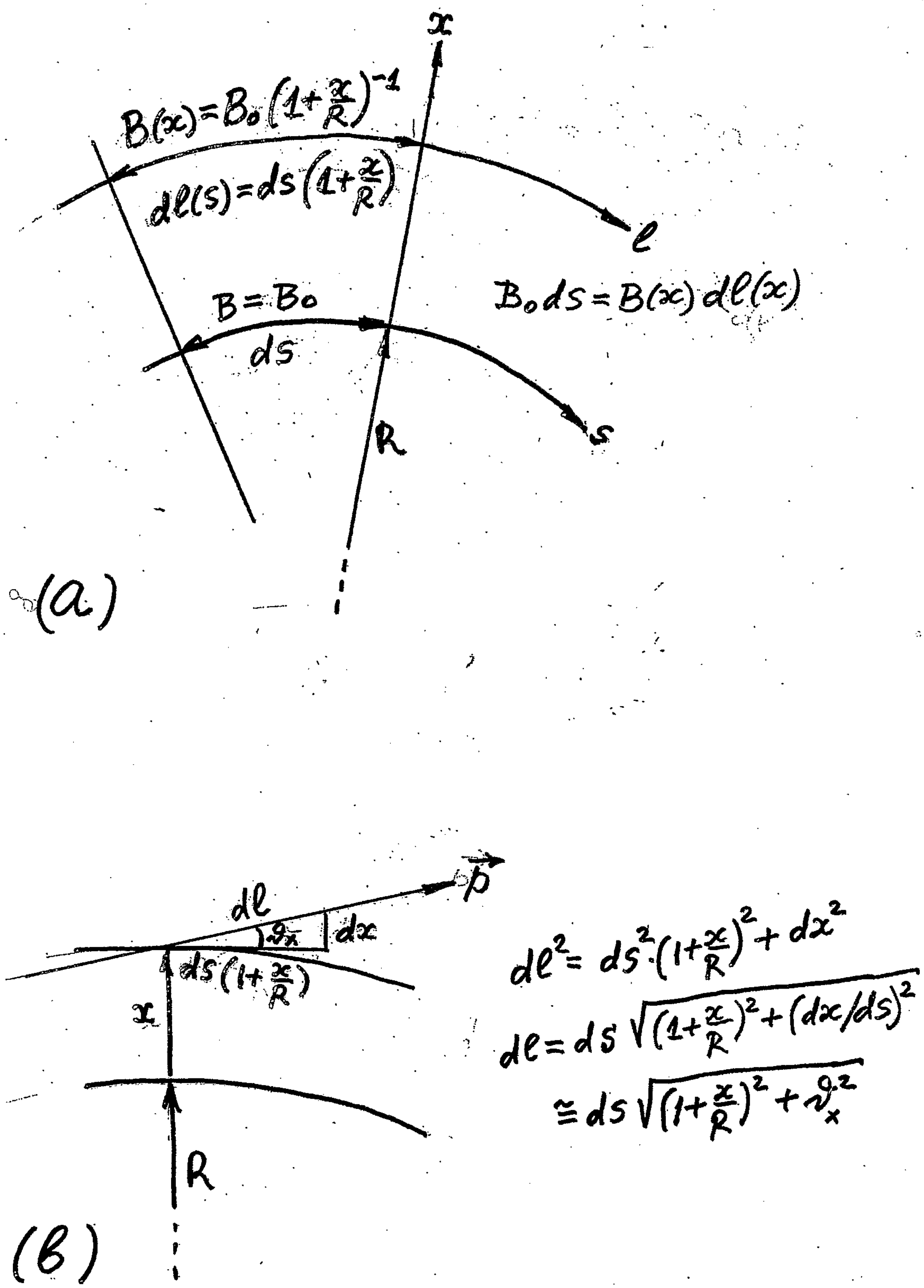
For $y \neq 0$, keeping only terms linear and quadratic in y,

$$B_V = \frac{B_0 R_0}{R}\left[1 - \frac{1}{2}\left(\frac{y}{R}\right)^2\right], \text{ inside BE,} \tag{3.3}$$

$$B_R = -\frac{B_0 R_0}{R^2} y, \text{ inside BE.} \tag{3.4}$$

This choice is rather arbitrary and needs to be compared in future with alternative choices. Different choices produce slightly different factors a_1, a_2, a_3 in formulas (1.3), (1.5), (1.6). The advantage of our choice of BE fields is that for particles with the ideal momentum, condition $\omega_a = 0$ holds for any x (but not for any dx/ds) in the central plane of the BE. The magnetic field index for field ((3.3)-(3.4), $n = -R(\partial B/\partial R)/B = 1$, equals 1 for every x in the central plane. If the particle is moving in the central plane, y=0, and in parallel to the ideal orbit, x=constant, then in all approximations there are no focusing forces either from E or from B fields because the path length, $dl = ds(1+x/R)$, is going up while the fields are going down with x *as* $1/(1+x/R)$, see Fig. 4a. In this case, the fields (3.1), (3.2), averaged over the particle trajectory inside BE's, always equal E_0 and B_0 independently of x, so there are no violations of condition (1.1) either.

However, perturbed trajectories are not parallel to the central axis. Correspondingly, there exist effects violating (1.1) and proportional to $\vartheta_x^2 \equiv (dx/ds)^2$. We will take them into account. (We will neglect only the average effects of acceleration in the horizontal E_R-field, which are also proportional to ϑ_x^2.) Nonzero $(dx/ds)^2$ and $(dy/ds)^2$ play the major role in the generalized horizontal and vertical pitch effects considered in the following sections. In particular, the lengthening of the trajectories due to x and y oscillations, see Fig. 4b



Figs. 4 a, b. The effects of trajectory lengthening.

our sextupoles (in order to cancel the off-momentum violation of (1.1)) is α_2, and we will show how it can be used.

We need to calculate all $\Delta p/p$ and $(\Delta p/p)^2$ terms violating (1.1). First of all, there exists a factor before E_R equal to $-(e/m)[|a|+m^2/p^2]\beta$ which directly depends on momentum. This coefficient produces

$$(\Delta\omega_a)_{coeff} = \frac{e}{m}|a|B\left\{\frac{1}{\gamma^2}\left(\frac{2E}{|a|\beta B}-1\right)\frac{\Delta p}{p} - \left[\frac{E}{|a|\beta\gamma^4 B}+\frac{3\beta^2}{2\gamma^2}\left(\frac{2E}{|a|\beta B}-1\right)\right]\left(\frac{\Delta p}{p}\right)^2\right\}\frac{L_{BE}}{L}, \quad (4.7)$$

and we know from (4.6) that the term $(\Delta p/p)$ here contains a non-oscillating part proportional to $(\Delta p/p)^2$. Below we will add to (4.7) more terms linear and quadratic in $\Delta p/p$ describing the field perturbations met by moving particles, and will investigate the meaning of $\langle\Delta p/p\rangle$. (The terms not connected to $\Delta p/p$ are considered in the next section.)

The next step is to analyze all effects following from the perturbations of the closed orbit due to $\Delta p/p$, $(\Delta p/p)^2$:

$$x(s) = x_\beta(s) + \tilde{D}(p,s)\frac{\Delta p}{p} + d(s)\left(\frac{\Delta p}{p}\right)^2. \quad (4.8)$$

Here, by definition, $d(s)$ depends only on sextupole fields, so $\alpha_2 = \langle d/R\rangle$. (But we can calculate α_2 without actual calculation of $d(s)$.) Our dispersion function, $\tilde{D}(p,s)$, is different from the usual $D(s)$,which does not depend on p . In fact, $\tilde{D}(p,s)$ is the (slightly approximate) solution of equation (17) in the Handbook [1] (on p.50), which does not take into account sextupoles. That equation is

Eq. (17) of [1]: $$D''(p,s) + \left(\frac{1}{R_0^2}+\frac{\partial B/\partial x}{BR_0}\right)\frac{p_0}{p}D(p,s) = \frac{1}{R_0}\frac{p_0}{p} + \frac{D(p,s)}{R_0^2}\frac{\Delta p}{p}. \quad (4.9)$$

The equation for the usual $D(s)$, which leads to our formulas (3.20)-(3.23), (3.33), corresponds to $p = p_0$ and $\Delta p = 0$ in (4.9). $\tilde{D}(p,s)$ is therefore the solution of (4.9), taking into account the next approximation in $\Delta p/p$. The last term in (4.9) is already proportional to $\Delta p/p$; so, with a very small error, we can substitute $\alpha_0 \equiv \langle D(s)/R_0\rangle$ for $D(p,s)/R_0$. Now, remember that in our BE sections,

$$\frac{1}{R_0^2}+\frac{\partial B/\partial x}{BR_0} = 0, \quad \frac{1}{R_0}\neq 0, \quad \text{BE sections.} \quad (4.10)$$

Finally, using (4.11) and the designed definition of the focal length, see (3.14), we get

$$\left(\frac{\langle \Delta B_V \rangle}{B_V}\right)_{p,quad} = \frac{L_{BE}}{L}\left(1+\alpha_0 \frac{\Delta p}{p}\right)\frac{\Delta p}{p}, \tag{4.24}$$

$$\left(\Delta \omega_a\right)_{p,quad} = \frac{e}{m}|d|B\frac{L_{BE}}{L}\left(1+\alpha_0 \frac{\Delta p}{p}\right)\frac{\Delta p}{p}. \tag{4.25}$$

Now, gathering all contributions together, using the formulas and numbers represented here, ($L_{BE}/L = 0.7547$, relativistic $\gamma^2 = 1.1765$,. $\alpha_0 = 0.03327$, etc.), we have:

$$(\Delta\omega_a)_p = \frac{e}{m}|d|B_V\frac{L_{BE}}{L}\left\{\left(1+\frac{1}{\gamma^2}\left(\frac{2E}{|a|\beta B}-1\right)\right)\left\langle\frac{\Delta p}{p}\right\rangle+\left[0.0164+\alpha_0-\left(\frac{E}{|a|\beta\gamma^4 B}+\frac{3\beta^2}{2\gamma^2}\left(\frac{2E}{|a|\beta B}-1\right)\right)\right]\left\langle\left(\frac{\Delta p}{p}\right)^2\right\rangle\right\}$$

(4.29)

4. Calculation and correction of $a_3(\Delta p/p)^2$.

As noted, the quadratic perturbations (1.3), (1.5), (1.6),

$$\Delta\omega_a = \Delta\omega_{a1} + \Delta\omega_{a2} + \Delta\omega_{a3} \equiv a_1 A_x^2 + a_2 A_y^2 + a_3(\Delta p/p)^2, \tag{4.1}$$

cannot be separated from the perturbation linear in $\Delta p/p$ when sychrotron stabilization holds. What the synchrotron oscillations actually stabilize is the average (in time) period of particle revolutions, T=L/v. If the relevant ring parameters are constant in time, then on the average (in time) all individual T's are the same. This means,

$$\left\langle \frac{\Delta(L/v)}{L_0/v_0} \right\rangle = \left\langle \frac{\Delta L}{L_0} - \frac{\Delta v}{v_0} - \frac{\Delta L}{L_0}\frac{\Delta v}{v_0} + \left(\frac{\Delta v}{v_0}\right)^2 + \ldots \right\rangle = 0. \tag{4.2}$$

This is the only feature of synchrotron oscillations needed for our purpose.

(From now on, we will omit indices "0" if this will not lead to ambiguities.) In this section we consider the case $x_\beta = y_\beta \equiv 0$. In such a case, we have (ignoring cubic and higher-order effects):

$$\left\langle \frac{\Delta L}{L} \right\rangle = \alpha_0 \frac{\Delta p}{p} + (\alpha_1 + \alpha_2 + \alpha_{D'})\left(\frac{\Delta p}{p}\right)^2, \tag{4.3}$$

$$\left\langle \frac{\Delta v}{v} \right\rangle = \frac{1}{\gamma^2}\left\langle \frac{\Delta p}{p} \right\rangle - \frac{3}{2}\frac{\beta^2}{\gamma^2}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle, \tag{4.4}$$

$$\left\langle \frac{\Delta L}{L}\frac{\Delta v}{v} \right\rangle = \frac{\alpha_0}{\gamma^2}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle. \tag{4.5}$$

The physical difference between second-order compaction factors α_1, α_2 and $\alpha_{D'}$ will be explained shortly. The usual compaction factor α_0 is given in (3.23) above. We see that in order to satisfy (4.2), the individual equilibrium of $\Delta p/p$ is shifted,

$$\left\langle \frac{\Delta p}{p} \right\rangle = \frac{1/\gamma^4 + 3\beta^2/2\gamma^2 - \alpha_0/\gamma^2 + \alpha_{D'} + \alpha_1 + \alpha_2}{1/\gamma^2 - \alpha_0}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle. \tag{4.6}$$

(There are more quadratic terms on the right side of (4.6) in the full expression for $\langle \Delta p/p \rangle$, if we take into account free betatron oscillations. These terms can be considered independently of $\langle (\Delta p/p)^2 \rangle$, as will be done in the following section.) The only factor here that can be changed by

$$\Delta L = L - L_0 = \int_0^{L_0} ds\sqrt{(1+x/R)^2 + (dx/ds)^2 + (dy/ds)^2} - L_0, \tag{3.5}$$

is the biggest contribution to a_i's. From (3.5), in the second-order approximation,

$$\frac{\Delta L}{L} = \left\langle \frac{x}{R} \right\rangle + \frac{1}{2}\left\langle \vartheta_x^2 \right\rangle + \frac{1}{2}\left\langle \vartheta_y^2 \right\rangle. \tag{3.6}$$

(3.6) is a purely geometrical effect. With respect to betatron oscillations, $\left\langle \vartheta_x^2 \right\rangle$ is proportional to A_x^2, and $\left\langle \vartheta_y^2 \right\rangle$ to A_y^2. In the synchrotron region, $\vartheta_x^2 = (D'(s))^2(\Delta p/p)^2$. Due to various nonlinear terms in the betatron equations, $\langle x/R \rangle$ itself depends on second-order effects,

$$\left\langle \frac{x}{R} \right\rangle = (\alpha_0 + \alpha_1 \frac{\Delta p}{p})\frac{\Delta p}{p} + \alpha_2 \left(\frac{\Delta p}{p}\right)^2 + q_x A_x^2 + q_y A_y^2, \tag{3.7}$$

The meaning of α_0, α_1, α_2, q_x, q_y will be clear later in this Note. (3.7) is a purely betatron dynamics effect. In (3.6) and (3.7), $\langle\ \rangle$ means averaging in time high frequency betatron oscillations, while $\Delta p/p$ is considered approximately constant in time. Then (3.6), (3.7) go into the equations for slow synchrotron oscillations. In the next section we will show that $\langle \Delta p/p \rangle$, being averaged over synchrotron oscillations, is shifted by all kinds of quadratic terms from its linear equilibrium $\Delta p/p = 0$, so

$$\left\langle \frac{\Delta p}{p} \right\rangle \propto \text{individual quadratic terms.} \tag{3.8}$$

Comparison of the $\left(\frac{\Delta p}{p}\right)^2$ effect and the trajectory lengthening.

$$\Delta \omega_a \sim \left(\frac{\Delta p}{p}\right)^2 + \left(\frac{\Delta L}{L}\right)_{betatron}$$

(if they are not cancelled).

$$\boxed{\left(\frac{\Delta p}{p}\right)^2 \sim 10^{-6}}$$

$$\left(\frac{\Delta L}{L}\right)_{bet} \sim \frac{1}{2}\vartheta_{x,y}^2 \equiv \frac{1}{2}\left(\frac{dx}{ds}\right)^2 \quad \text{or} \quad \frac{1}{2}\left(\frac{dy}{ds}\right)^2.$$

$$x \cong x_0 \cos\left(\frac{2\pi\nu_x}{L}s + \varphi\right)$$

$$dx/ds = -x_0 \frac{\nu_x}{R} \sin\left(\frac{2\pi\nu_x}{L}s + \varphi\right)$$

$$\frac{1}{2}\vartheta_x^2 \approx \underline{\frac{1}{2}\left(\frac{\nu_x}{R}\right)^2 x_0^2} \sin^2\left(\frac{\nu_x}{R}s + \varphi\right)$$

$\nu_x \sim 5$

$x_0 \sim 2.5$ cm

$R = 15$ m

$$\frac{1}{2}\left(\frac{2.5\times 10^{-2}}{15}\right)^2 \cdot 25 =$$

$$\cong 0.3\times 10^{-4} \gg 10^{-6}$$

$$(\Delta\omega_a)_p = a_3\left(\frac{\Delta p}{p}\right)^2, \quad a_3 = \frac{e}{m}|a|B_V[0.9881 + 1.9437\alpha_2]. \tag{4.30}$$

To eliminate a_3, we need

$$\alpha_2 = -\sum \frac{B_i'' D_i^3 l_{si}}{2BR}\frac{1}{L} = -0.51. \tag{4.31}$$

For an estimation, assume that we can use only 10 sextupoles placed next to + quads, $l_s = 0.5m$, $D^+ = 0.8921m$, B=0.2095T, R=15m, L=124.89m. Then we need $B'' = 90m^{-2}$. ANL uses sextupoles with $B'' = 415m^{-2}$ (for Advanced Proton Source, see [1], p. 443, table 2).

5. Calculation and correction of the $a_1 A_x^2$ and $a_2 A_y^2$ terms.

We now discuss the (generalized) horizontal and vertical pitch effects. The horizontal pitch effect has not been previously noted. (In my EDM Note #10, I considered only cases $x_\beta = dx_\beta / ds = 0$, that is $A_x^2 = 0$.)

There are four physically different effects leading to the dependence of $\Delta\omega_a$ on A_x^2, A_y^2.

1. The first effect is the result of the combination of trajectory lengthening, $\Delta L/L$, due to $\vartheta_x^2 \equiv (dx_\beta / ds)^2$ and $\vartheta_y^2 \equiv (dy/ds)^2$, see formula (3.6), and the synchrotron stability leading to dependence of $\langle \Delta p/p \rangle$ on this lengthening. The effect can be calculated immediately.

$$\left(\frac{\Delta L}{L}\right)_{x_\beta', y'} = \frac{1}{2}\left(\langle\vartheta_x^2\rangle + \langle\vartheta_y^2\rangle\right) = \frac{1}{4}\frac{1}{s}\int_0^s ds' \left(\gamma_x(s')A_x^2 + \gamma_y(s')A_y^2\right), \tag{5.1}$$

where $\gamma = (1+\alpha^2)/\beta$ is one of the three Courant-Snyder parameters. In (5.1) $s \to \infty$. Formula (5.1) follows from formulas (2), (3) of [1], p.49,

$$x(s) = A_x\sqrt{\beta_x(s)}\cos(\psi_x(s)+\delta_x), \quad x'(s) = -\frac{A_x}{\sqrt{\beta_x(s)}}\left[\alpha_x(s)\cos(\psi_x(s)+\delta_x) + \sin(\psi_x(s)+\delta_x)\right] \tag{5.2}$$

and analogously for y. A, δ are constants,

$$A^2 = x_{\max}^2(s)/\beta(s) = (x_{\max}^2(s))_{\max}/\beta_{\max}. \tag{5.3}$$

So if, for example, $(x_{\max}(s))_{\max}$=2.5cm, and $\beta_{\max}$=6.5m, then A^2=0.9654$\times 10^{-4}$m. In our lattice, $\beta_{\max} = \beta^+$.

In (5.1), γ_x =constant between quadrupoles (but not inside quadrupoles), see Edwards and Syphers [6], p.97. We have (with γ-values given in (3.19), (3.32) above),

4. We obviously need to compensate effects proportional to $A_{x,y}^2$ because with $A_{x,y}^2 \sim 10^{-4}$ and $(\Delta p/p)^2 \sim 10^{-6}$, the violation of condition (1.1) by betatron oscillations is more than one order larger than the violation by the momentum spread. So the fourth effect is the effect of sextupoles used for these compensations. When the particle performing betatron oscillations passes sextupole #i periodically, it periodically gets a horizontal angle deflection equal to

$$(\theta_x)_i = -\frac{B_i'\left(x^2 - y^2\right)_i l_{si}}{2BR}. \tag{5.15}$$

On the average,

$$\left\langle (\theta_x)_i \right\rangle = -\frac{B''(\beta_x A_x^2 - \beta_y A_y^2)_i l_{si}}{4BR}. \tag{5.16}$$

According to [1], p.263, formula (2), this periodic deflection shifts the horizontal equilibrium, so

$$\left(\frac{\Delta L}{L}\right)_{x'_\beta} = \left(\frac{\Delta L}{L}\right)_{x'_\beta BE} + \left(\frac{\Delta L}{L}\right)_{x'_\beta I} = \frac{A_x^2}{4}\left[\gamma_{x,BE}\frac{L_{BE}}{L} + \gamma_{x,I}\frac{L_I}{L}\right] = 0.2134A_x^2. \tag{5.4}$$

This goes into $\langle \Delta p/p \rangle$,

$$\left\langle \frac{\Delta p}{p} \right\rangle_{x'_\beta} = \frac{(\Delta L/L)_{x'_\beta}}{1/\gamma^2 - \alpha_0} = 0.2613A_x^2. \tag{5.5}$$

This, in turn, goes into

$$(\Delta\omega_a)_{x'_\beta} = \frac{e}{m}|d|B_V\frac{L_{BE}}{L}\left[1 + \frac{1}{\gamma^2}\left(\frac{2E}{|d|\beta B} - 1\right)\right]\left\langle \frac{\Delta p}{p} \right\rangle = \frac{e}{m}|a|B_V \cdot 0.4147A_x^2. \tag{5.6}$$

$\gamma_y(s)$ has exactly the same value as γ_x=1.0444 in the straight sections. It is slightly different, and not exactly constant, in the BE's, because vertical oscillations are focused there (the field index n=1). In the linear approximation, in which γ_y is defined,

$$y'' + \frac{1}{R^2}y = 0, \quad \text{inside BE's, linear approximation.} \tag{5.7}$$

We get

$$\left(\frac{\Delta L}{L}\right)_{y'} = 0.2139A_y^2, \quad (\Delta\omega_a)_{y'} = \frac{e}{m}|d|B_V \cdot 0.4157A_y^2. \tag{5.8}$$

2. The second effect also depends on betatron angles, ϑ_x, ϑ_y. Due to these angles, magnetic and electric fields met by a spin passing a BE section differ from their designed values. With respect to the electric field, the average vector product $\vec{v} \times \vec{E}$ is changed. (Such an effect was not dangerous in our (g-2) ring only because, in the g-2 experiment, the equilibrium electric field equalled zero.) As for the magnetic field, the effect is the usual F. Farley pitch. The best way to understand both electric and magnetic field perturbations in this case is to analyze the J.D. Jackson formula (11.171), [7], p.550:

$$\frac{d}{dt}\left(\frac{\vec{v}}{v}\cdot\vec{s}\right) = -\frac{e}{mc}\vec{s}_{perp}\cdot\left[\left(\frac{g-2}{2}\right)\frac{\vec{v}}{v}\times\vec{B} + \left(\frac{g\beta}{2} - \frac{1}{\beta}\right)\vec{E}\right], \tag{5.9}$$

where $\vec{v}$ is velocity, $\beta = v/c$, $\vec{s}$ is the rest frame spin, $\vec{s}_{perp}$ is a part of the spin vector perpendicular to $\vec{v}$. We consider the case of $\vec{B} = \vec{B}_V$ perpendicular, and $\vec{E} = \vec{E}_R$ parallel to the ideal orbit plane. $\vec{v} \times \vec{B}$ is also parallel to the plane; therefore, only the component of $\vec{s}_{perp}$ lying in the plane contributes to rotation of the spin relative to velocity. It is easy to see, with a little algebra, that if $\vec{v}$ is also lying in the plane and is perpendicular to $\vec{E}$, and condition (1.1) is

$$\left(\frac{\Delta L}{L}\right)_q = -\sum \frac{B_i' D_i (\beta_{xi} A_x^2 - \beta_{yi} A_y^2) l_{si}}{4BRL}. \tag{5.17}$$

Because of the synchrotron stability, (5.17) leads to the shift of the momentum equilibrium,

$$\left\langle \frac{\Delta p}{p} \right\rangle_q = \frac{(\Delta L / L)_q}{1/\gamma^2 - \alpha_0} = -\frac{1}{(1/\gamma^2 - \alpha_0)} \sum \frac{B_i' D_i (\beta_{xi} A_x^2 - \beta_{yi} A_y^2) l_{si}}{4BRL}. \tag{5.18}$$

And, finally, this leads to the corresponding violation of condition (1.1). Using the term proportional to $\langle \Delta p / p \rangle$ in our formula (4.26), we get

$$(\Delta \omega_a)_q = -0.9719 \sum \frac{B_i'' D_i \beta_{xi} l_{si}}{2BRL} A_x^2 + 0.9719 \sum \frac{B_i' D_i \beta_{yi} l_{si}}{2BRL} A_y^2. \tag{5.19}$$

After gathering all betatron terms, we have

$$(\Delta \omega_a)_{xq} \equiv a_1 A_x^2 = \left(0.564 - 0.9719 \sum \frac{B_k' D_k \beta_{xk} l_{sk}}{2BRL} \right) A_x^2, \tag{5.20}$$

$$(\Delta \omega_a)_{yk} \equiv a_2 A_y^2 = \left(0.2629 + 0.9719 \sum \frac{B_k'' D_k \beta_{yk} l_{sk}}{2BRL} \right) A_y^2. \tag{5.21}$$

6. Conclusion.

Thus, the situation is the following. To cancel a_1, a_2, a_3, we need to satisfy three conditions:

$$1. \sum \frac{B_k' D_k \beta_{xk} l_{sk}}{2BRL} = 0.58, \qquad a_1 = 0.$$

$$2. \sum \frac{B_k' D_k \beta_{yk} l_{sk}}{2BRL} = -0.27, \qquad a_2 = 0.$$

$$3. \sum \frac{B_k' D_k^3 l_{sk}}{2BRL} = 0.51, \qquad a_3 = 0.$$

It is instructive to compare these conditions with the conditions for the betatron chromaticity cancellation:

$$\sum \frac{B_k' D_k \beta_{xk} l_{sk}}{2BRL} \approx 0.43, \qquad \xi_x = 0.$$

$$\sum \frac{B_k' D_k \beta_{yk} l_{sk}}{2BRL} \approx -0.43 \qquad \xi_y = 0.$$

We see, first, that when we satisfy our conditions 1 and 2, we simultaneously reduce the x-chromaticity to -26%, and the y-chromaticity to 37% of their original magnitudes. And second, If $(\frac{\Delta p}{p})^2 = 10^{-6}$, $(x)_{max} = (y)_{max} = 2.5$cm, then the coherent time ~ 10 s can be achieved by satisfying 1 and 2 only, with accuracy $\sim 1\%$. To get coherent time ~ 1 min, it is sufficient to squeeze $(\frac{\Delta p}{p})$ by factor three, and to satisfy 1 and 2 with accuracy 0.15%.

A small effect of acceleration in the E_R field. (Remark)

$$\mathcal{E} = \sqrt{m^2c^4 + p_L^2c^2 + p_x^2c^2 + p_y^2c^2} + \Delta\mathcal{E}_{E_R}$$

$$\frac{d\mathcal{E}}{dt} = e v_x E_R \cong e \dot{x} E_0 \left(1 - \frac{x}{R}\right)$$

$$\Delta\mathcal{E}_{E_R} = \underbrace{e E_0 x}_{\text{oscillates}} - \frac{1}{2} e \frac{x^2}{R} E_0 \longrightarrow -\frac{1}{2} e E_0 \frac{x^2}{R}$$

$$\mathcal{E} = \underbrace{\sqrt{m^2c^4 + p_L^2c^2}}_{\mathcal{E}_0} \underbrace{\left\{ 1 + \frac{1}{2} \frac{p_x^2c^2}{m^2c^4 + p_L^2c^2} \right\}}_{\mathcal{E}_x/\mathcal{E}_0} - \frac{1}{2} e E_0 \frac{x^2}{R}$$

$$\mathcal{E}_x = \frac{p_L^2c^2}{2\mathcal{E}_0} \vartheta_x^2$$

$$\Delta\mathcal{E}_{E_R} = -\frac{1}{4} e E_0 \frac{x_0^2}{R}$$

$$\langle \mathcal{E}_x \rangle = \frac{1}{2} \frac{p_L^2c^2}{2\mathcal{E}_0} \left\langle \left(\frac{\nu_x x_0}{R} \right)^2 \right\rangle$$

$$\boxed{\frac{\Delta\mathcal{E}_{E_R}}{\mathcal{E}_x} \sim 10^{-2}}$$

$$e E_0 R = (4 \times 10^{-3}) \frac{\text{GeV}}{m} \times 15\, m = 0.06\ \text{GeV}$$

$$\left(\frac{R}{x_0}\right)^2 \mathcal{E}_x \sim 6.5\ \text{GeV}$$